The topological interpretation of the core group of a surface in S^4

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Abstract. We give a topological interpretation of the core group invariant of a surface embedded in S^4 [F-R, Ro]. We show that the group is isomorphic to the free product of the fundamental group of the double branch cover of S^4 with the surface as a branched set, and the infinite cyclic group. We present a generalization for unoriented surfaces, for other cyclic branched covers, and other codimension two embeddings of manifolds in spheres.

It is shown in [Ya-1] (compare [Ka-1, C-S-2, Ya-2]) that the fundamental group of the complement of an oriented surface, M, in S^4 allows a Wirtinger type presentation. In [Ro] the core group invariant of M was introduced (following [F-R, Joy]) and the topological interpretation of the group was promised. In the first section we give this interpretation following the result of Wada's [Wa] for classical knots and the proof of Wada result presented in [Pr]. In the second section we generalize our results to unoriented surfaces, and codimension two embeddings of closed n-manifolds in S^{n+2}.

1 Surfaces in S^4

Let M be an oriented surface embedded in $S^4 = R^4 \cup \{\infty\}$. Let $p : R^4 \to R^3$ be a projection such that the restriction to M, $p|_M$ is a general position map. Define the lower decker set ("invisible" set) [C-S-1, C-S-2] as $A = \{x \in M \mid \exists y \in M, \ p(x) = p(y) \ and \ x \ is \ the \ lower \ point, \ that \ is \ x \ is \ further \ than \ y \ in \ the \ direction \ of \ the \ projection\}$. The set A separates M into regions ("visibility" regions), that is arc connected components of $M - A$.

Theorem 1.1 ([Ya-1]) *The fundamental group of $S^4 - M$ ($\Pi_M = \pi_1(S^4 - M)$) has the following Wirtinger type presentation. Generators of the group,*

$x_1, x_2, ..., x_n$, correspond to regions of M. Relations of the group correspond to double point arcs and are of the form $x_i x_p x_i^{-1} x_q^{-1}$ or $x_i x_p^{-1} x_i^{-1} x_q$, depending on the orientations of $p(M)$ at double point arcs. Here x_i corresponds to the higher region of the projection and x_p, x_q to lower regions.

Similarly to the group Π_M, we define the core group, G_M of a surface M in S^4.

Definition 1.2 ([Ro]) *The core group, G_M of a surface M in S^4, has the following, quandle type, presentation. Generators of the group, $y_1, y_2, ..., y_n$, correspond to regions of M. Relations of the group correspond to double point arcs and are of the form $y_i y_p^{-1} y_i y_q^{-1}$. Here y_i corresponds to the higher region of the projection and y_p, y_q to lower regions.*

The operation $y_p * y_i = y_i y_p^{-1} y_i$ defines a quandle.
We will interpret topologically the core group as follows. It is the analogue of the Wada result for classical links [Wa].

Theorem 1.3 *The core group, G_M of a surface M in S^4, is isomorphic to the free product of the fundamental group of the branched double cover of S^4 with branching set M, and the infinite cyclic group. That is $G_M = \pi_1(M^{(2)}) * Z$ where $M^{(2)}$ is the considered double branched cover.*

Below we will prove a generalization of the theorem for any cyclic branched cover, following the exposition of the classical case in [Pr].

Let $f : M^{(k)} \to S^4$ be a cyclic k fold branched covering with a branch set M. More precisely $M^{(k)}$ is defined as follows: Let V_M be a tubular neighborhood of M in S^4. It has a structure of a 2-disc bundle (over M). For a point $x \in M$ the boundary of the disk D_x of the bundle is called a meridian of M (at x), denoted by μ_x. It has the natural orientation for oriented M. The first homology group of $S^4 - M$ is freely generated by meridians of M (one for each connected component of M). Thus, we have a k-fold cyclic covering of $S^4 - M$ given by an epimorphism $H_1(S^4 - M) \to Z_k$ which sends μ_x to 1. This covering can be uniquely extended to the cyclic k fold branched covering with a branch set M. Namely, we define the cover on each (meridian) disk D_x as given by the function $p : D'_x \to D_x$, $p(z) = z^k$ where z is a complex coordinate of the disks.

Let $\Pi_M^k = \pi_1(M^{(k)})$.

Proposition 1.4 *Consider the epimorphism $\hat{g}_k : \pi_1(S^4 - M) \to Z_k$ given by $\hat{g}_k(x_i) = 1$ for any i. Then*

$$\Pi_M^k = ker(\hat{g}_k)/(x_i)^k.$$

Proof: Consider a regular neighborhood of M in S^4 and its normal disk bundle. Each disk in the bundle is covered by a disk (the center being a branch point). As noted before the covering can be described by the map $z \to z^k$ where z is a complex coordinate of the disk. In the Wirtinger presentation of the fundamental group, generators correspond to some meridians (that is, boundaries of disks of the normal bundle).
Consider the unbranched k-covering of $S^4 - M$ obtained by removing the branch set. The epimorphism $\hat{g} : \pi_1(S^4 - M) \to Z_k$ corresponding to the covering sends each meridian to $1 \in Z_k$, $(\hat{g}(\mu_x) = 1)$. The fundamental group of the unbranched cover is equal to $ker\hat{g}$. To get a branched cover from the unbranched one we have to fill the cover of every meridian. On algebraic level we add relations $\mu_x^k = 0$. Because every meridian is conjugated to a generator x_i, the proposition is proved. $\square$

Theorem 1.5 $\Pi_M^{(k)} * \underbrace{Z * ... * Z}_{k-1} = \Pi_{M \sqcup S^2}^{(k)}$ *has the following quandle type de-scription (in the disjoint sum $M \sqcup S^2$, S^2 is the unknotted component that can be separated by a 3-sphere from M).*
To every (visible) region there correspond $k-1$ generators $\tau^j(y_i)$ $(0 \le j < k-1$) and to each double point arc correspond $k-1$ relations $\tau^j(y_p \tau(y_i)\tau(y_q^{-1})y_i^{-1})$ or $\tau^j(y_i\tau(y_p)\tau(y_i^{-1})y_q^{-1})$ depending on a local orientation.

Lemma 1.6 *Let $F_{n+1} = \{x_1, x_2, ..., x_{n+1} \mid \}$ be a free group of $n+1$ generators, $x_1, x_2, ..., x_{n+1}$. Consider the epimorphism $g_k : F_{n+1} \to Z_k$ such that $g_k(x_i) = 1$ for any i. Let $F^{(k)} = ker(g_k)$, and let $\bar{F}^{(k)} = F^{(k)}/(x_i^k)$. Let $\tau : F_{n+1} \to F_{n+1}$ be an automorphism, defined by $\tau(x_i) = x_{n+1}x_ix_{n+1}^{-1}$ and $y_i = x_ix_{n+1}^{-1}$. Then*

   (i) $F^{(k)}$ is a free group of $nk + 1$ generators : x_{n+1}^k and $\tau^j(y_i)$ for $0 \le j \le k - 1$, $1 \le i \le n$.

   (ii) $\bar{F}^{(k)}$ is a free group on $n(k - 1)$ generators $\tau^j(y_i)$ for $0 \le j < k - 1$, $1 \le i \le n$.
   Furthermore $y_i\tau(y_i)...\tau^{k-1}(y_i) = 1$ for any i.
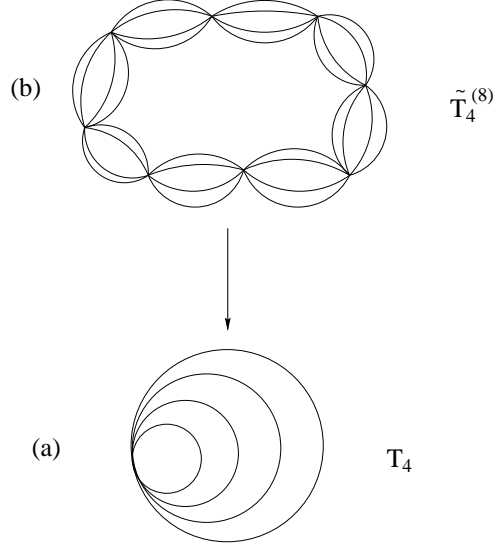
(b)

$\tilde{T}_4^{(8)}$

(a)

T_4

Fig. 1 (The covering $\tilde{T}_4^{(8)} \to T_4$)

Proof:

(i) $F_{n+1} = \pi_1(T_{n+1})$ where T_{n+1} is a graph of one vertex and $n+1$ loops (Fig.1(a)). $g_k : F_{n+1} \to Z_k$ determines a k-fold covering space $\tilde{T}_{n+1}^{(k)}$ of the graph T_{n+1} as show in Fig.1(b). Of course, the Euler characteristic $\chi(T_{n+1}) = -n$ and $\chi(\tilde{T}_{n+1}^{(k)}) = -kn$. Thus, $\pi_1(\tilde{T}_{n+1}^{(k)})$ is freely generated by $nk+1$ generators. We can easily identify free generators of $\pi_1(\tilde{T}_{n+1}^{(k)}) = ker g_k$; in particular, $g_k(x_{n+1}^k) = g_k(\tau^j(y_i)) = 0$ for $1 \le i \le n$ and $0 \le j \le k-1$.

(ii) $y_i \tau(y_i) \tau^2(y_i)...\tau^{k-1}(y_i) = (x_i x_{n+1}^{-1})(x_{n+1} x_i x_{n+1}^{-2})...(x_{n+1}^{k-1} x_i x_{n+1}^{-k}) = x_i^k x_{n+1}^{-k}$. Thus, in $F_{n+1}^{(k)}/(x_{n+1}^k)$ we have $y_1...\tau^{k-1}(y_i) = x_i^k$ and furthermore
$\bar{F}_{n+1}^{(k)} = \{F_{n+1}^{(k)} \mid x_{n+1}^k, x_i^k ; 1 \le i \le n\}=$
$\{x_{n+1}^k, \tau^j(y_i) \mid x_{n+1}^k, y_i\tau(y_i)...\tau^{k-1}(y_i), 1 \le i \le n, 0 \le j \le k-1\} =$
$= \{\tau^j(y_i) \mid y_i\tau(y_i)...\tau^{k-1}(y_i), 1 \le i \le n, 0 \le j \le k-1\} = \{\tau^j(y_i) \mid 1 \le i \le n, 0 \le j < k-1\}$ as required.

□

We will use an extension of Lemma 1.6 for a group with Wirtinger type relations.
Let $G = \{F_n \mid r_1, ..., r_m\} = \{x_1, ..., x_n \mid r_1, ..., r_m\}$ where any relation is of the

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form $r = x_i x_p^\epsilon x_i^{-1} x_q^{-\epsilon}$ for $\epsilon = \pm 1$, and $1 \le i, p, q \le n$

We have $G * Z = \{x_1, ..., x_{n+1} | r_1, ..., r_m\}$. The epimorphism $g_k : F_{n+1} \to Z_k$ yields the epimorphism $\hat{g}_k : G * Z \to Z_k$. The epimorphism $\hat{g}_k$ is well defined because $g_k(r_i) = 0$. We use Lemma 1.6 to find a presentation of $G^{(k)} = ker\hat{g}_k$ and $\bar{G}^{(k)} = G^{(k)}/(x_i^k)$.

Lemma 1.7 (i) $G^{(k)} = \{F^{(k)} | \tau^j(r_s), 1 \le s \le m, 0 \le j \le k - 1\}$.

(ii) $\bar{G}^{(k)} = \{\bar{F}^{(k)} \mid \tau^j(r_s), 1 \le s \le m, 0 \le j < k - 1\} = \{\tau^j(y_i) | \tau^j(r_s), 1 \le s \le m, 0 \le j < k - 1, 1 \le i \le n\}$ where $\tau^j(x_i x_p x_i^{-1} x_q^{-1}) = \tau^j(y_i \tau(y_p) \tau(y_i^{-1}) y_q^{-1})$ and $\tau^j(x_i x_p^{-1} x_i^{-1} x_q) = \tau^j(y_i y_p^{-1} \tau^{-1}(y_i^{-1}) \tau^{-1}(y_q))$.

Proof:

(i) Because $r_s \in ker g_k$ then the relations of $ker\hat{g}_k$ are of the form $w r_i w^{-1}$ for $w \in F_{n+1}$. We observe that if $g_k(w) = u$ ($0 \le u \le k - 1$) then $w = x_{n+1}^u w'$ where $w' \in ker g_k$. Therefore relations of $G^{(k)} = ker\hat{g}_k$ are of the form $\tau^u(r_s)$ for $0 \le u \le k - 1, 1 \le s \le m$. These yield the presentation of $G^{(k)}$. We have: $x_i x_p x_i^{-1} x_q^{-1} =$
$(x_i x_{n+1}^{-1})(x_{n+1}(x_p x_{n+1}^{-1}) x_{n+1}^{-1})(x_{n+1}(x_{n+1} x_i^{-1}) x_{n+1}^{-1})(x_{n+1} x_q^{-1}) =$
$y_i \tau(y_p) \tau(y_i^{-1}) y_q^{-1}$
$x_i x_p^{-1} x_i^{-1} x_q =$
$(x_i x_{n+1}^{-1})(x_{n+1} x_p^{-1})(x_i^{-1} x_{n+1})(x_{n+1}^{-1} x_q) = y_i y_p^{-1}(\tau^{-1}(y_i))^{-1} \tau^{-1}(y_q)$

(ii) Adding relations $(x_i)^k$ reduces $F^{(k)}$ to $\bar{F}^{(k)}$ and $G^{(k)}$ to $\bar{G}^{(k)}$ so $\bar{G}^{(k)} = \{\bar{F}^{(k)} | \tau^j(r_s), 1 \le s \le m, 0 \le j \le k - 1\}$ We can eliminate the relation $\tau^{k-1}(r_s)$ (expressing it using other relations). We use the identity $y_i \tau(y_i)...\tau^{k-1}(y_i) = 1$ in $\bar{G}^{(k)}$ (in particular $\tau^k(y_i) = y_i$). We can write our relations $\tau^u(y_i \tau(y_p) \tau(y_i^{-1}) y_q^{-1})$ as $\tau^{u+1}(y_p y_i^{-1}) = \tau^u(y_i^{-1} y_q)$. We assume it holds for $0 \le u < k - 1$ and we will see that $\tau^{k-1}(r_s) = 1$ as well:

$$\tau^{k-1}(y_i \tau(y_p) \tau(y_i^{-1}) y_q^{-1}) = \tau^{k-1}(y_i) \tau^k(y_p) \tau^k(y_i^{-1}) \tau^{k-1}(y_q^{-1}) =$$
$$= \tau^{k-1}(y_i) y_p y_i^{-1} \tau^{k-1}(y_q^{-1}) = \tau^{k-1}(y_i) y_p (y_i^{-1} y_q) \tau(y_q)...\tau^{k-2}(y_q) =$$
$$= \tau^{k-1}(y_i) y_p \tau(y_p)(\tau(y_i^{-1})\tau(y_q))\tau^2(y_q)...\tau^{k-2}(y_q) =$$
$$= \tau^{k-1}(y_i) y_p \tau(y_p) \tau^2(y_p)(\tau^2(y_i^{-1})\tau^2(y_q))\tau^3(y_q)...\tau^{k-2}(y_q) =$$
$$= \tau^{k-1}(y_i) y_p \tau(y_p) \tau^2(y_p) \tau^3(y_p)...\tau^{k-1}(y_p) \tau^{k-1}(y_i^{-1}) = 1$$

as required.

□

Theorem 1.5 follows from Proposition 1.4 and Lemma 1.7. In particular, Theorem 1.3 follows because for $k = 2$ we have $y_i\tau(y_i) = 1$ so $\tau(y_i) = y_i^{-1}$. In the next section, we show that the recent result of Kamada [Ka-2] allows us to extend Theorem 1.5 to any closed oriented n-manifold in S^{n+2}. Similarly Theorem 1.3 can be generalized to any closed unoriented n-manifold in S^{n+2}.

2 Higher dimensional case: $M^n \subset S^{n+2}$.

We can extend our results to the case of a closed n-manifold M in S^{n+2} (not necessary oriented or connected). First we need an existence of the Wirtinger type presentation of the fundamental group of $S^{n+2} - M$ for oriented M, described in [Ka-2] and its variant for unoriented M.

Theorem 2.1 *([Kamada])*
Let $p : R^{n+2} \to R^{n+1}$ be a projection and M be in general position with respect to the projection. We can think that the base point is very high above R^{n+1} (say at $\infty \in S^{n+2}$). We have $n - 1$ dimensional strata being the closure of the (invisible) set $\{x \in M| \ \exists y \in M, \ p(x) = p(y) \ and \ x \ is \ the \ lower \ point\}$. These strata cut M into n-dimensional regions ("visibility regions"). Let Π_M denote the fundamental group of $S^{n+2} - M$ ($\Pi_M = \pi_1(S^{n+2} - M)$).

(1) Assume that M is oriented.
Then Π_M has the following Wirtinger type presentation. Generators of the group, $x_1, x_2, ..., x_n$, correspond to regions of M (and can be visualize by joining the base point b to any meridian of the region; meridians are oriented so elements of Π_M are well defined). Relations of the group correspond to double point strata and are of the form $x_i x_p x_i^{-1} x_q^{-1}$ or $x_i x_p^{-1} x_i^{-1} x_q$, depending on orientations of $p(M)$ at double point strata. Here x_i corresponds to the higher region of the projection and x_p, x_q to lower regions.

(2) Consider now M not necessary oriented. Let generators, $x_1, x_2, ..., x_n$ be chosen as before (one for each region). Because a region may be unorientable we have to make choices here. However, modulo relations $x_i^2 = 1$ (for $1 \leq i \leq n$), the group has Wirtinger type presentation. That is: $\Pi_M/(x_i^2)$ has a presentation as in (1) with additional relations $x_i^2 = 1$.

In the unoriented case meridians have no preferred orientation so only the branched double cover is uniquely defined, so we can interpret the core group of the embedded unoriented codimension two submanifold of S^{n+2}.

Kamada's theorem allows us to generalize Theorems 1.3 and 1.5. The proof, as before, bases on Lemma 1.7 ($k = 2$ yields an unoriented case), so we omit it.

Let $p : R^{n+2} \to R^{n+1}$ be a projection and M be an n-dimensional closed submanifold of R^{n+2} (and $S^{n+2} = R^{n+2} \cup \infty$) being in general position with respect to the projection. Using notation of Theorem 2.1 we have.

Theorem 2.2 *Let $M^{(k)}$ denote the cyclic branched regular k-covering of S^{n+2} with an oriented manifold M as a branched set, and $\Pi_M^{(k)}$ its fundamental group. Then $\Pi_M^{(k)} * \underbrace{Z * ... * Z}_{k-1} = \Pi_{M \sqcup S^n}^{(k)}$ has the following quandle type description.*
To every region corresponds $k - 1$ generators $\tau^j(y_i)$ ($0 \le j < k - 1$) and to each double point strata correspond $k - 1$ relations $\tau^j(y_p \tau(y_i) \tau(y_q^{-1}) y_i^{-1})$ or $\tau^j(y_i \tau(y_p) \tau(y_i^{-1}) y_q^{-1})$ depending on a local orientation.

Theorem 2.3 *Let $M^{(2)}$ denote the double branched regular 2-covering of S^{n+2} with an unoriented (possibly not orientable) manifold M as a branched set. Then $\pi_1(M^{(2)}) * Z = \Pi_{M \sqcup S^n}^{(2)}$ has the following core (quandle type) description.*
Generators of the group, $y_1, y_2, ..., y_n$, correspond to regions of M. Relations of the group correspond to double point strata and are of the form $y_i y_p^{-1} y_i y_q^{-1}$.

In the case of a 3-manifold in S^5, one can check invariantness of our groups, defined combinatorically, using the Roseman moves [Ros-1].

3   Acknowledgment.

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